Generalized observers and velocity measurements in General Relativity

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Abstract

To resolve some unphysical interpretations related to velocity measurements by static observers, we discuss the use of generalized observer sets, give a prescription for defining the speed of test particles relative to these observers, and show that, for any locally inertial frame, the speed of a freely falling material particle is always less than the speed of light at the Schwarzschild black hole surface.

1 Introduction

The radial motion of a test particle falling in a Schwarzschild black hole was treated by several authors [1, p.298], [2, p.93],[3, pp.19,20],[4, p.342], [5, 6] who reached the same conclusion that the particle velocity v approaches the light velocity as the test particle approaches the surface of the black hole, namely the locus $r = 2m$ (with a suitable choice of units), also known as the event horizon or Schwarzschild radius. All these authors have in common the use of observers whose worldlines are the integral curves of a hypersurface orthogonal Killing vector field, that is, static observers (called shell observers in [7, p.2-33]) and, as such, at rest with respect to the mass creating the gravitational field. For example, Zel'dovich and Novikov say that the velocity they use "has direct physical significance. It is the velocity measured by an observer who is at rest $(r, \theta, \phi, \text{constant})$ at the point the particle is passing." [2, p.93]. The particle's motion here is referred to the Schwarzschild coordinate system in which the line element takes the form

$$d\tau^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{1}$$

in geometric units $(c = G = 1)$.

Following along the same lines, Frolov and Novikov recently [3, pp.19,20] add that "The physical velocity v measured by an observer who is at rest in the Schwarzschild reference frame situated in the neighborhood of the freely moving body is

$$v = \frac{dx}{d\tau} = \left[\frac{g_{11}}{|g_{00}|}\right]^{1/2} \frac{dr}{dt} = \pm \frac{[E^2 - 1 + r_g/r]^{1/2}}{E}. \tag{2}$$

If the falling body approaches $r_g(= 2m)$, the physical velocity $v = dx/d\tau$ constantly increases: $v \to 1$ as $r \to r_g$." In Eq.(2), $E = (1 - 2m/r)dt/d\tau$ is a constant of motion which we may interpret for timelike geodesics as representing the total energy per unit rest mass of a particle following the geodesic in question, relative to a static observer at infinity [8, p.139].

In their very well known textbook [4, p.342], Shapiro and Teukolsky also produce a similar statement: "... the particle is observed by a local static observer at r to approach the event horizon along a *radial* geodesic at the *speed of light* ..."

All these statements have contributed to the wrong and widespread view [9] that makes its way into the literature [10], of a test particle approaching the event horizon at the speed of light *for all observers*, and not as a limiting process for a static observer sitting at r, as $r \to 2m$. At the first sight, this view seems quite logical since we expect the particle to cross the black hole surface in a finite proper time. And if one accepts that a particle has the speed of light with respect to a static observer (at $r = 2m$), using locally the velocity composition law from special relativity, he (or she) concludes that the particle has the same speed of light with respect to all observers. This is certainly something that conflicts with the physical observation that, in a vacuum, no material particle travels as fast as light. This has been very nicely explained by Janis who established that the test particle does indeed cross $r = 2m$ with a speed less than the speed of light [11]. Here we take a similar view, and go one step further in obtaining a general expression for geodesic radial observers in terms of the constants of motion of both observer and test particle.

In Sec. 2 we discuss the different mathematical status of coordinate charts and reference frames, and compare this present attitude with the early days when it was quite common the use of 'curvilinear four-dimensional coordinate system' and 'frame of reference' interchangeably. In Sec. 3, we review some standard results and definitions of reference frames and observer sets, and give a prescription for the speed at some space-time point relative to generalized observers at that point. We find the speed of any material particle to be strictly less than the speed of light. In Sec. 4 we apply this general prescription to the Schwarzschild field and reproduce Eq.(2) for shell observers, then we recall that there is a static limit, and we obtain an expression, valid at $r = 2m$, for the test particle's square speed as a function of the constants of motion of the observer and the particle, when both follow radially inward geodesics. Finally, we give a brief discussion of the results.

2 Coordinate Systems and Reference Frames

One of the underlying principles of general relativity is the freedom of choice of coordinates in the mathematical description of laws and physical quantities. Indeed, the outcome of

physical measurements depends, in general, on the reference frame, that is, on the 'state of motion' of the observer, but cannot depend on the coordinate system chosen, which may be completely arbitrary and should be selected for convenience in the intermediate calculation. Of course, certain coordinates may be preferred over other coordinates in the sense that they are simpler or better adapted to the symmetries of the gravitational field under consideration.

The association of an arbitrary coordinate system with an arbitrary frame of reference became standard in the literature for many decades after the advent of general relativity. Then, it was quite common the use of 'curvilinear four-dimensional coordinate system' and 'frame of reference' interchangeably, as Bergmann explains in [12, pp.158,159]: "...we have always represented frames of reference by coordinate systems ...". This point is even stressed when he adds: "The equivalence of all frames of reference must be represented by the equivalence of all coordinate systems."

In our discussion, we find necessary to make a distinction between "reference frames" and "coordinate systems". By a reference frame we shall mean an observer set by which measurements are directly made. For example, a set of radially moving geodesic observers would comprise a frame of reference. On the other hand, a coordinate system refers to a set of numbers assigned to each point in the space-time manifold. That is, we follow a common view in which "...coordinates charts are today given a quite different mathematical status than that of the frames of reference" [13, pp.419-434], whereas they were previously considered suitable for a given reference frame rather than for an extended view of the whole manifold.

In Newtonian physics a reference frame is an imagined extension of a rigid body and a clock. We can then choose different geometrical coordinate systems or charts (Cartesian, spherical, etc.) for the same frame. For example, the earth determines a rigid frame throughout all space, consisting of all points which remain at rest relative to the earth and to each other. One can associate an orthogonal Cartesian coordinate system with such a reference frame in many ways, by choosing three mutually orthogonal planes and using the coordinates x, y, z as the measured distances from these planes. As soon as a time coordinate t is defined one is ready to label any physical event. It should be stressed

that this choice of coordinates presupposes that the geometry in such a frame is Euclidean.

But what is precisely a reference frame in general relativity? And how does it differ from a special relativity inertial frame? To build a physical reference frame in general relativity it is necessary to replace the rigid body by a fluid [14, p.268] or a cloud of point particles that move without collisions but otherwise arbitrarily. In more mathematical terms, one can define [3, p.627] a *reference frame* as a future-pointing, timelike congruence, that is, a three-parameter family of curves $x^a(\lambda, y^i)$, where λ is a parameter along the curve and y^i is a set of parameters that 'labels' the curves, such that one and only one curve of the family passes each point. If specific parameters λ and y^i are chosen on the congruence, we define a *coordinate system*. Of course, this choice is not unique. Thus, in general, a given reference frame can give rise to more than one associated coordinate system. And a particular coordinate system may or may not be associated with an obvious reference frame.

Let us define an *observer* in a space-time as a material particle parameterized by proper time [15, p.36]. An observer field (or reference frame) on a space-time M is a future-pointing, timelike unit vector field. Observers enjoy sending and receiving messages, and keep close track of their proper time. In special relativity a single geodesic observer can impose his (or her) proper time on the entire Minkowski space-time, but in general relativity, "a single observer is so local that only cooperation between observers gives sufficient information"[16, p.52], that is, a whole family of observers is needed for analogous results.

3 Generalized Observers

Given the four-velocity field, u, of an observer set $\mathcal{O}$ we parametrize the world lines of $\mathcal{O}$ with the proper time measured by a clock comoving with each observer ("wrist-watch time"), so that we have $g_{ab}u^a u^b = u^a u_a = -1$; u^a is a geodesic reference frame iff in addition it is parallel propagated along itself: $\nabla_u u = 0$. The integral curves of u are called observers in u (or u-observers, for short). All observers in a geodesic reference frame are freely falling.

An observer field u on M is *stationary* provided that exists a smooth function $f > 0$ on M such that $fu = \xi$ is a Killing vector field, that is, the Lie derivative of the metric with respect to the vector field ξ vanishes

$$
\begin{aligned}
L_\xi g_{ab} &\equiv \xi^c \partial_c g_{ab} + g_{cb}\partial_a \xi^c + g_{ac}\partial_b \xi^c \\
&= \nabla_a \xi_b + \nabla_b \xi_a = 0.
\end{aligned}
\tag{3}
$$

If the one-form corresponding to ξ is also hypersurface orthogonal

$$
\xi_a \equiv \lambda \partial_a \phi,
$$

where λ and ϕ are two scalar fields, then each u-observer is *static* (i.e., $u^\perp$ is integrable). In this case the integral manifolds $u^\perp$ are three-dimensional, spacelike submanifolds that are isometric under the flow and constitute a common rest space for the u-observers.

Let us consider a test particle given by its 4-velocity vector field $t^a = dx^a/d\tau$. We can decompose t^a into a timelike component and a spacelike component by applying a time-projection tensor, $-(u_a u_b)$, and a space-projection tensor, $h_{ab} \equiv g_{ab} + u_a u_b$:

$$
t^a_\parallel = -u^a u_b t^b, \quad t^a_\perp = h^a_b t^b.
\tag{4}
$$

One can easily verify that $t^a_\parallel$ e $t^a_\perp$ are timelike and spacelike, respectively. Then we rewrite the space-time distance ds^2 between two events x^a and $x^a + dx^a$ of the test particle's wordline as

$$
ds^2 = -(u_a dx^a)^2 + h_{ab}dx^a dx^b = -dt_*^2 + d\ell_*^2.
\tag{5}
$$

That is, separation of time and space is always possible infinitesimally, and an (instantaneous) observer in x^a, with four-velocity u^a, measures between the two events x^a and $x^a + dx^a$ of the particle's wordline a proper space and proper time given respectively by

$$
d\ell_* = (h_{ab}dx^a dx^b)^{1/2},
\tag{6}
$$

and

$$
dt_* = -u_a dx^a.
\tag{7}
$$

The asterisks in Eqs.(6) and (7) denote that the quantities so indicated are not, in general, exact differentials. The minus sign in Eq.(7) gives dt_* the same sense as dx^0.

There is a natural way for an u-observer to define the speed of any particle with four-velocity t^a as it passes through an event $p \in M$. As the observer has instantaneous information at p that allows him (or her) to break up the tangent space $T_p(M)$ at p into time t (parallel to u) and space $u^\perp$, he (or she) will measure

$$v^2 = (\frac{d\ell_*}{dt_*})^2 = \frac{(g_{ab} + u_a u_b)t^a t^b}{(u_a t^a)^2}, \tag{8}$$

for the square of the speed of the particle at p, which can be written as,

$$v^2 = 1 - \frac{1}{(u_a t^a)^2}. \tag{9}$$

Whatever is the particle's four-velocity, t^a, one can always write it as

$$t^a = t^a_\parallel + t^a_\perp = \lambda u^a + \ell^a, \quad \text{where} \quad \ell_a u^a = 0. \tag{10}$$

Since t^a should be timelike, $t^a t_a = -\lambda^2 + |\ell|^2 < 0$ (notice that $|\ell|^2 = \ell^a \ell_a = h_{ab}t^a t^b = (t^a)_\perp (t_a)_\perp$), and since both t^a and u^a are future-pointing, $\lambda = -u_a t^a > 0$, and $|\ell| < \lambda = (|\ell|^2 + 1)^{1/2}$. From this last equation and from (9) one immediately concludes that under these conditions $v^2 < 1$. The number λ represents the instantaneous rate at which the observer's time is increasing relative to the particle's time, and $|\ell|$ is the rate at which arc length $d\ell_*$ in $u^a_\perp$ is increasing relative to the particle's time, that is,

$$\lambda = \frac{dt_*}{d\tau}, \quad |\ell| = \frac{d\ell_*}{d\tau}. \tag{11}$$

Thus the u-observer measures the speed of the t-particle at event p as

$$v = \frac{d\ell_*}{dt_*} = \frac{d\ell_*/d\tau}{dt_*/d\tau} = \frac{|\ell|}{\lambda} < 1. \tag{12}$$

Notice that, from Eq. (12), $v = 1$ iff the t-particle is lightlike ($t^a t_a = 0$); otherwise, for timelike particles, $v < 1$.

4 The Schwarzschild Field Case

Having dealt with this problem in a very general way and proved that the velocity v of any massive particle with respect to any physical observer is always smaller than the velocity of light: $v < 1$, let us apply these ideas to the Schwarzschild gravitational field and find a general prescription for evaluating v when both the particle and the observer are geodesic.

4.1 Geodesic test particle

Let us suppose that our test particle follows a radially ingoing geodesic in a Schwarzschild field. Its geodesic equation of motion is the Euler equation for the Lagrangian $2L = g_{ab}\dot{x}^a\dot{x}^b$, which is given by

$$2L = -\alpha\dot{t}^2 + \alpha^{-1}\dot{r}^2, \tag{13}$$

where $\alpha = -g_{00} = g_{11}^{-1} = 1 - 2m/r$, for the Schwarzschild metric Eq.(1) with $\theta = const.$ and $\varphi = const.$, and the dot, as usual, denotes differentiation with respect to proper time. Along the orbit

$$2L = -1, \tag{14}$$

for the particle's proper time is given by

$$d\tau^2 = \alpha dt^2 - \alpha^{-1}dr^2. \tag{15}$$

From this we could also write

$$d\tau^2 = \alpha dt^2(1 - v^2), \tag{16}$$

where

$$v^2 = \frac{1}{\alpha^2}\left(\frac{dr}{dt}\right)^2, \tag{17}$$

is, accordingly to Eq.(8), the velocity of the particle with respect to a static observer ($r = constant$); i.e. while the particle travels a proper distance $\alpha^{-1/2}dr$ the observer measure a proper time given by $\alpha^{1/2}dt$.

Eq. (13) shows that t is a cyclic coordinate, and

$$-\frac{\partial L}{\partial \dot{t}} = (1 - 2m/r)\dot{t} = const. =: E, \tag{18}$$

is the constant of motion along the geodesic associated with the Killing vector field $\partial/\partial t$; that is, if the particle's 4-velocity t^a is geodesic, $\nabla_t t = 0$, then: $\nabla_t[g(t, \partial/\partial t)] = 0$, which equally implies Eq. (18).

Inserting Eqs.(17) and (18) into Eq.(14) gives

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - \alpha, \tag{19}$$

and from this we obtain

$$E^2 = \frac{\alpha}{1 - v^2} = \frac{1 - 2m/R}{1 - v_0^2}, \tag{20}$$

where (R, v_0) are initial conditions; R is the radial coordinate at which the fall begins, and v_0 is the initial velocity.

Now, from Eqs.(18) and (19) we obtain the components of the 4-velocity t^a of a radially ingoing geodesic particle

$$t^a = \left(\frac{E}{\alpha}, -\sqrt{E^2 - \alpha}, 0, 0 \right), \tag{21}$$

written in terms of its constant of motion E.

4.2 Static limit

In Landau and Lifchitz [1, p.250] the velocity is measured in terms of proper time, as determined by clocks synchronized along the trajectory of the particle, as they say. Their prescription leads to the following expression

$$v^2 = \left(g_{00} + g_{01} \frac{dx^1}{dx^0} \right)^{-2} \left(g_{01}^2 - g_{00} g_{11} \right) \left(\frac{dx^1}{dx^0} \right)^2, \tag{22}$$

for the square of the velocity of a radially moving particle.

We have seen earlier, that there is a natural way for the observer u^a to measure the speed of any particle with four-velocity t^a as it passes through an event $p \in M$, which is coordinate free, and given by Eq.(9). For a static observer the 4-velocity has the following components

$$u^a u_a = -1 \Rightarrow u_a = (-g_{00})^{-1/2} g_{a0},$$

and for the test particle, its tangent vector to radially inward, timelike geodesics may be written as

$$t^a = \left(\frac{dx^0}{d\tau}, \frac{dx^1}{d\tau}, 0, 0 \right).$$

Inserting these last two 4-vector components in Eq.(9) leads to Eq.(22), which must be understood as a specialization of Eq.(9) for static observers.

When applied to the (geodesic) radial motion of a free falling particle, Eq.(22) leads to Eq.(17) which can be rewritten as

$$v = \left[1 - \frac{1 - 2m/r}{E^2} \right]^{1/2}, \tag{23}$$

which is equivalent to Eq.(2). In the case when $E = 1$, corresponding to $R = \infty$ or $v_0 = 0$, it reduces to

$$v = \left(\frac{2m}{r} \right)^{1/2}, \tag{24}$$

which coincides with the Newtonian expression

For either expression, v approaches the speed of light at the event horizon $(r = 2m)$ and they seem to to predict faster-than-light speeds inside the black hole[17]. It is easily seen that

$$\lim_{r \to 2m} v = 1 \quad \text{and} \quad \lim_{r \to 0} v = \infty, \tag{25}$$

for both Eqs. (23) and (24).

Taken at face value the previous statements would imply that the particle's trajectory should become lightlike in the limit $r \to 2m$. However, as the trajectory can be continued through the event horizon, it seems clear that it must remain timelike there, otherwise we had to conclude that the particle's velocity would overcome the light speed as its worldline becomes spacelike.

Since this is an unacceptable result and we know that the Schwarzschild coordinate system is not suitable for describing the manifold at $r = 2m$ it is rather tempting to blame the coordinate system for this malfunction. But we should ask first, could it be possible to find a coordinate system that does not have this defect? The answer is obviously no, since the result is independent of the choice of coordinates, as we have proved in the third section of this paper, and as will be even clearer at end of this section. Indeed, even if we use a coordinate system that has no difficulties at $r = 2m$, like the advanced Eddington-Finkelstein coordinates, we would still end up with the same result $v^2 \to 1$ as $r \to 2m$.

We can easily see this by introducing the Eddington-Finkelstein metric [18, p.828],

$$ds^2 = - \left(1 - \frac{2m}{r} \right) dw^2 + 2dwdr + r^2 \left(d\theta^2 + \sin^2 \theta d\varphi^2 \right), \tag{26}$$

11

where

$$w(t, r) = t + r + 2m \ln \left| \frac{r - 2m}{2m} \right|, \tag{27}$$

in Eq.(22), valid for static observers. Then, at $r = 2m$, where $g_{00} = 0$, we obtain $v^2 = 1$.

Thus the real issue here is the choice of frame not the choice of coordinates. For instance, the process of synchronizing clocks, used by Landau and Lifchitz[1, p.250], involves the determination of simultaneous events at different spatial locations, which is a frame-dependent prescription.

Notice also that an observer cannot stay at rest in a Schwarzschild field at $r = 2m$, where $g_{ab} u^a u^b = 0$, for he (or she) cannot have there a timelike four-velocity field tangent to its worldline. This means that only a photon can stay at rest at $r = 2m$, and with respect to this "photon-frame" all particles have $v^2 = 1$, as it should be expected.

Another argument that could be given, although it is closely related to the later discussion, is provided by the study of the acceleration of a static observer in a Schwarzschild field [19]. Whereas a static Newtonian observer is considered to be at rest in its own proper "inertial frame", in general relativity an observer at rest is not geodesic and is accelerated. To make it clear(er) let's evaluate the acceleration of a static observer in a spherically symmetric and static gravitational field. Starting with its four-acceleration field components in Schwarzschild coordinates,

$$a^c = u^c{}_{;b} u^b = u^c{}_{,b} + u^a \Gamma^c_{ab} = g^{ca} g_{00,a} g_{00}^{-1}/2,$$

one finds that all nonradial components vanish

$$a^0 = a^\theta = a^\varphi = 0, \text{ and } a^r = -\frac{1}{2} g_{00,1} = \frac{m}{r^2}. \tag{28}$$

However, a^r, which is the radially inward acceleration as calculated using Newtonian gravity, is a coordinate-dependent quantity and it is not a scalar field. The invariant acceleration magnitude that we require is

$$a = (a^c a_c)^{1/2} = -(g_{11})^{1/2} g_{00,1}/2.$$

For the Schwarzschild field this gives

$$a = \frac{m}{r^2} \left(1 - \frac{2m}{r} \right)^{-1/2}. \tag{29}$$

The factor $(g_{11})^{1/2}$, by which the GR and Newtonian accelerations differ, can be neglected in most cases $(r \gg 2m)$, such as apply e.g. on the surface of planets or even on the surfaces of normal stars. For instance, for the Sun $(2m/R)_\odot = 4.233 \times 10^{-6}$. But on the surface of a neutron star $(g_{11})^{1/2}$ may exceed unity by a very large factor, and for a black hole

$$a \to \infty, \quad \text{as} \quad r \to 2m.$$

"It follows that a 'particle' at rest in the space at $r = 2m$ would have to be a photon" [20, p.149]. This makes it very clear that we should define a *static limit* of a black hole, that is, the boundary of the region of space-time in which the observer can remain at rest relative to any observer in the asymptotically flat space-time. In plain words, as any observer must follow a timelike worldline, the static limit is given by

$$g_{00}(r) = 0 \quad \text{(static limit)}.$$

This emphasizes the point that one cannot use expressions like (17) or (22) at the surface $r = 2m$. In other words, there is no observer at rest on that surface. As Taylor and Wheeler put it in their recent textbook [7, p.3-15]: "Shell–and shell observers–cannot exist inside the horizon or even *at* the horizon, where the spherical shells experiences infinite stresses."

4.3   Radial observers

Considering that the particle and the observer are both in free fall (inward, timelike geodesics), we can use Eq.(21) and write respectively

$$t^a = \left(\frac{E_1}{\alpha}, -\sqrt{E_1^2 - \alpha}, 0, 0 \right), \quad u^a = \left(\frac{E_2}{\alpha}, -\sqrt{E_2^2 - \alpha}, 0, 0 \right). \tag{30}$$

Then inserting these into Eq.(9) the following expression is obtained,

$$v^2 = 1 - \frac{\alpha^2}{E_1^2 E_2^2 \left[1 - \sqrt{1 - \frac{\alpha}{E_1^2}} \sqrt{1 - \frac{\alpha}{E_2^2}} \right]^2}, \tag{31}$$

and since $\alpha = 1 - 2m/r$, it follows that
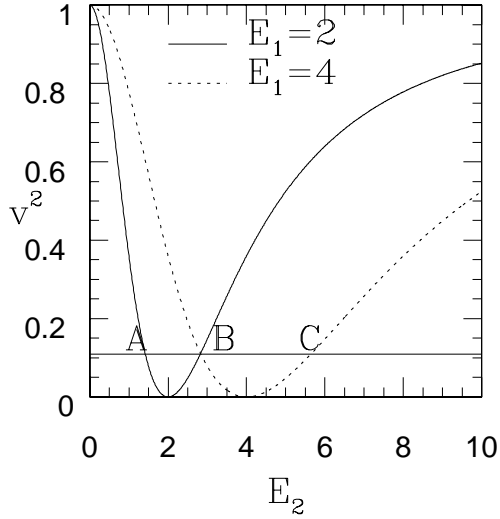
$$\lim_{r \to 2m} v^2 = 1 - 0/0.$$

Figure 1: Test particle's square speed v^2 at $r = 2m$ as function of the observer's constant of motion, for two cases (particle with $E_1 = 2$ and $E_1 = 4$).

However, $(1 - \alpha/E^2)^{1/2}$ may be expanded if $r \approx 2m$ since $\alpha/E^2 \ll 1$,

$$\left(1 - \frac{\alpha}{E^2}\right)^{1/2} = 1 - \frac{1}{2}\frac{\alpha}{E^2} - \frac{1}{8}\frac{\alpha^2}{E^4} + O\left(\frac{\alpha}{E^2}\right)^3.$$

This leads to

$$v^2 = 1 - \frac{\alpha^2}{E_1^2 E_2^2}\left[\left(\frac{\alpha(E_1^2 + E_2^2)}{2E_1^2 E_2^2} + O\left(\frac{\alpha}{E^2}\right)^2\right)^2\right]^{-1} \tag{32}$$

and we now obtain an exact expression for the velocity at $r = 2m$,

$$v^2(r = 2m) = 1 - \frac{4E_2^2 E_1^2}{(E_2^2 + E_1^2)^2} \tag{33}$$

which shows (see figure1) that the value of the velocity at $r = 2m$ is smaller than 1 unless either E_1 or E_2 are zero or infinity. In particular, when $E_1 = E_2$, we see that $v^2(r = 2m) = 0$. This means that particle and observer have the same initial conditions at some space-time point p with $r > 2m$, and from that event onwards they are both on the same local inertial frame.

Notice that for each particle there are 2 observers who measure the same value of v^2. For example observers A and B for one particle and B and C for the other. Notice also

that the constant of motion of the particle is always in between the two values of E_2 of those observers. This means, picturing the first particle and the observers A and B all free falling, that when they all meet at $r = 2m$, the particle reaches the observer A with the same velocity that the observer B reaches the particle with(and the same for B,C and the other particle). In other words, the value of v is positive in the first branch of the plots (before the minimum) and negative in the second branch.

Considering now the 2 particles, we notice there is an observer (B), with E in between the values of the particles' constants of motion, who measures the same value of v^2 for both particles. Once again this means that at $r = 2m$, B catches one of the particles (the first one) with the same speed with which it is caught by the other.

In fact, owing to symmetry between test particle and observer, both situations are equivalent. And we could say that figure 1 refers to the observer's square speed v^2 at $r = 2m$ as function of the particle's constant of motion.

Now, we highlight the fact that only in the limits $E \to 0$ and $E \to \infty$ (from what we have just seen, it is indifferent if E refers to the particle or to the observer), we obtain $v = 1$. In these cases we conclude the hypothetical 'observer' (or test particle) is in a "photon-frame". In fact referring to Eq.(20) we see these two limits correspond to either $v_0 = 1$ or $R = 2m$. With respect to this (unphysical) frame all particles travel at the speed of light $v = 1$.

Expressions similar to Eq.(33) can be found [21, 22] for the velocity of a free falling particle in the Schwarzschild field, derived for diverse non-static observers.

As an example, let us consider a Kruskal observer, an observer which follows an orbit defined by

$$u^a = \left(\frac{dt'}{d\tau}, \frac{dx'}{d\tau} \right),$$

with $dx' = 0$, where (t', x') are the Kruskal coordinates.

For $r > 2m$, these coordinates (x', t') relate to the Schwarzschild ones by,

$$
\begin{cases}
x'^2 - t'^2 = \left(\frac{r-2m}{2m} \right) e^{r/2m} \\
\\
t' = \tanh \left(\frac{t}{4m} \right) x'
\end{cases}
\tag{34}
$$

and the metric takes the form [18, p.832],

$$ds^2 = \frac{32m^3}{r}e^{-r/2m}(-dt'^2 + dx'^2) + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{35}$$

From here we see that an observer which maintains the space-like coordinate x' constant, verifies,

$$\frac{32m^3}{re^{r/2m}}\left(\frac{dt'}{d\tau}\right)^2 = 1. \tag{36}$$

Differentiating Eq.(34) we get,

$$\frac{dr}{d\tau} = \frac{8m^2}{e^{r/2m}r}\left(x'\frac{dx'}{d\tau} - t'\frac{dt'}{d\tau}\right), \qquad \frac{dt}{d\tau} = \left(x'\frac{dt'}{d\tau} - t'\frac{dx'}{d\tau}\right)\frac{8m^2}{e^{r/2m}(r - 2m)}. \tag{37}$$

Using $dx' = 0$ and Eq.(36) we can write the following equation :

$$\left(1 - \frac{2m}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{2m}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 = \frac{2m(x^2 - t^2)}{e^{r/2m}(r - 2m)} = 1, \tag{38}$$

which shows this observer follows a radial trajectory.

Consider now a material particle along a radial ingoing geodesic. From Eq.(8), its velocity, measured by a Kruskal observer is

$$v = \frac{dx'}{dt'}, \tag{39}$$

since Eq.(35) is diagonal with $g_{x'x'} = g_{t't'}$, which is analogous to Eq.(17). Dividing one of the equations Eq.(37) by the other and solving for v we obtain,

$$v = \frac{1 + \tanh(t/4m)\frac{dt}{dr}(1 - 2m/r)}{\tanh(t/4m) + \frac{dt}{dr}(1 - 2m/r)}, \tag{40}$$

where dt and dr refer to the movement $t(r)$ of the particle.

We can now introduce the geodesic dt/dr followed by the particle, from Eq.(19) which we can also explicitly integrate to obtain an expression for $t(r)$ to substitute in Eq.(40). The details can be found in [22], where the behavior of the v against E plot was found to be identical to the one presented in this paper.

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5 Conclusions and Discussion

We have seen that the speed of any material particle following a radially inward geodesic is strictly less than 1 with respect to any physical (timelike) observer. We recalled that there is a limit for the use of static observers in a Schwarzschild field given by: $g_{00}(r) = 0$. Thus, we stress the point that one can only use static observers in the space-time region characterized by $r > 2m$. We found a formula for the physical velocity of a test particle in a radially inward, timelike geodesic, measured by an observer in free fall (which crosses the event horizon simultaneously with the particle) valid at $r = 2m$.

We conclude that all free falling observers crossing the black hole surface measure the speed of light ('standing still' photons at $r = 2m$) to be $v = 1$, and they measure the speed of any material particle to be strictly less than 1.

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